Lecture 25

11.5 - Alternating Series

The tests we dealt with so far only allow us to deal with series with all positive terms (we can use them to deal with series with all negative terms too, by factoring out the -1), but what do we do with series with both positive and negative terms?

Def: An alternating series is a series of the form $$\sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^n b_n$$
where $b_n > 0$ for all n.

Ex: $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{\pi}{3}\right)^{2n-1}}{(2n-1)!}$$

are alternating series.

Alternating Series Test

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $\left(\text{or } \sum_{n=1}^{\infty} (-1)^n b_n\right)$

satisfies (i) $b_n \geq b_{n+1}$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series converges.

(**Note**: The usual divergence test applies to alternating series, indeed any series.)

proof: $S_2 = b_1 - b_2 > 0$, $S_4 = S_2 + \underbrace{b_3 - b_4}_{>0} > S_2$, $\cdots$, $S_{2n+2} = S_{2n} + \underbrace{b_{2n+2} - b_{2n+1}}_{>0} > S_2$

So, $0 < S_2 < S_4 < S_6 < \cdots < S_{2n} < \cdots$ Moreover,

$S_{2n} = b_1 \underbrace{- b_2 + b_3}_{<0} \underbrace{- b_4 + b_5}_{<0} - \cdots - b_{2n} < b_1$

So, $S_2, S_4, \cdots, S_{2n}, \cdots$ is a bounded monotonic sequence, thus $\{S_{2n}\}$ converges, & let's say $\lim_{n\to\infty} S_{2n} = S$. The odd terms:

$S_{2n+1} = S_{2n} + b_{2n+1}$, satisfy $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = S + 0 = S$

Thus $\lim_{n\to\infty} S_n = S$, and so $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

$\square$

Ex: Determine whether the following series converge:

ⓐ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ⓑ $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$ ⓒ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$

ⓓ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{2n+3}$ ⓔ $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{2\pi}{n}\right)$

Sol:

ⓐ $b_n = \frac{1}{n}$. (i) $\frac{1}{n+1} = b_{n+1} \leq b_n = \frac{1}{n}$ ✓ (ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

convergent

ⓑ $b_n = \frac{3n-1}{2n+1}$ (ii) $\lim_{n\to\infty} b_n = \frac{3}{2} \neq 0$ So, the alternating series test fails.

In fact, since $\lim_{n\to\infty} (-1)^n \frac{3n-1}{2n+1}$ does not exist, by the divergence test, this series diverges.

ⓒ $b_n = \frac{1}{\ln(n+4)}$ (i) $\frac{1}{\ln(n+5)} = b_{n+1} \leq b_n = \frac{1}{\ln(n+4)}$ ✓ Since $\ln(n+5) \geq \ln(n+4)$.

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\ln(n+4)} = 0$ ✓ convergent

ⓓ $b_n = \frac{\sqrt{n}}{2n+3}$. (i) Hard to see by directly comparing b_n & b_{n+1}.

$b_n = f(n) = \frac{\sqrt{n}}{2n+3}$.

$$f'(n) = \frac{\frac{1}{2\sqrt{n}}(2n+3) - \sqrt{n}\,(2)}{(2n+3)^2} = \frac{\sqrt{n} + \frac{3}{2}\frac{1}{\sqrt{n}} - 2\sqrt{n}}{(2n+3)^2} = \frac{\frac{3}{2}\frac{1}{\sqrt{n}} - \sqrt{n}}{(2n+3)^2} < 0 \text{ for } n \geq 2$$

So, $b_{n+1} \leq b_n$ for $n \geq 2$

(ii) $\lim_{n\to\infty} \frac{\sqrt{n}}{2n+3} = 0$ convergent

ⓔ $b_n = \cos\left(\frac{2\pi}{n}\right)$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \cos\left(\frac{2\pi}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{2\pi}{n}\right) = \cos(0) = 1 \neq 0$

So, the alternating series test fails.

Moreover, since

$\lim_{n\to\infty} (-1)^n \cos\left(\frac{2\pi}{n}\right)$ does not exist, the series diverges

by the divergence test.

Estimating the Error

Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges to s. That is, the sequence of partial sums $s_n = b_1 - b_2 + b_3 - \cdots + (-1)^{n-1}b_n$ converges to s. Because of the way alternating series converge, it's pretty simple to estimate the sum and get an error bound, $|R_n| = |s - s_n|$

Alternating Series Estimation Theorem

If $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$, $b_n > 0$ such that

(i) $b_{n+1} < b_n$ for all n & (ii) $\lim_{n\to\infty} b_n = 0$

then $$|R_n| = |s - s_n| \leq b_{n+1}.$$

proof: $s_{2n} \leq s$ from the earlier proof

$$s = s_{2n} + b_{2n+1} \underbrace{- b_{2n+2} + b_{2n+3}}_{>0} \underbrace{- b_{2n+4} + b_{2n+5}}_{>0} \cdots < s_{2n} + b_{2n+1} = s_{2n+1}$$

So, $s_{2n} \leq s \leq s_{2n+1}$. Moreover, $s_{2n+1} = s_{2n-1} \underbrace{- b_{2n} + b_{2n+1}}_{<0} \leq s_{2n-1}$

So, $s_{2n} \leq s \leq s_{2n-1}$. Thus

$$|R_n| = |s - s_n| \leq |s_{n+1} - s_n| = |b_{n+1}| = b_{n+1}$$

$\square$

Ex: How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$$

do we have to add up to approximate the sum to within 10^{-5}?

Sol: To make calculations easier, note we can write

$$|R_n| = |s - s_n| \leq b_{n+1}$$

as

$$|R_{n-1}| = |s - s_{n-1}| \leq b_n$$

So, we want $|R_{n-1}| \leq 10^{-5}$, so let's solve

$$\frac{1}{n^6} = b_n = 10^{-5} \Rightarrow n^6 = 10^5 \Rightarrow n = 10^{5/6}$$

$10^{5/6} \approx 6.813$. So we should take $n = 7$.

This means $|R_{7-1}| = |R_6| \leq 10^{-5}$, so we only need to add the first 6 terms to be within the desired error.